

# Fermions without vierbeins in curved space-time

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(September 21, 2000)

A general formulation of spinor fields in Riemannian space-time is given without using vierbeins. The space-time dependence of the Dirac matrices required by the anticommutation relation  $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$  determines the spin connection. The action is invariant under any local spin base transformations in the 32 parameter group  $Gl(4, \mathbb{C})$  and not just under local Lorentz transformations. The Dirac equation and the energy-momentum tensor are computed from the action.

04.20.-q, 04.62.+v, 11.10.-z

## I. INTRODUCTION

The Dirac equation for spinor fields was formulated in curved space in 1929 by Fock and Ivanenko [1], who found it convenient to employ vierbeins. That method provides a particular solution for the space-time dependent Dirac matrices  $\gamma^\nu$  satisfying the anticommutation relation

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}I. \quad (1.1)$$

The description of fermions in curved space-time using vierbeins can be found in books by deWitt [2], Weinberg [3], Burell and Davies [4], and Wald [5]. It frequently implied, and sometimes explicitly stated, that fermionic fields cannot be formulated in curved space without the use of vierbeins. This paper will show that vierbeins are not necessary.

The reason for eschewing vierbeins is not mere novelty. Always using vierbeins to treat fermionic fields is rather like always using the Coulomb gauge to perform electrodynamics calculations in Minkowski space. One can get the right answers in Coulomb gauge. However it would be very puzzling if physics could only be done in that gauge because the Coulomb gauge condition only holds in one Lorentz frame, a requirement which is contrary to the principle that all inertial frames are equivalent.

The vierbeins are the gravitational analogue of the Coulomb gauge. At each point  $x$  in a curved space-time it is always possible to choose a set of four locally inertial coordinates  $\xi^a$ , which depend on  $x$ . The vierbeins are defined as the partial derivatives

$$e_\mu^a(x) = \frac{\partial \xi^a}{\partial x^\mu}. \quad (1.2)$$

Since coordinates that are inertial at one point will not be inertial at a nearby point, the vierbeins have the property that  $\partial_\nu e_\mu^a \neq \partial_\mu e_\nu^a$ . They automatically satisfy the two

relations  $e_\mu^a e_\nu^b \eta_{ab} = g_{\mu\nu}$  and  $e_\mu^a e_\nu^b g^{\mu\nu} = \eta^{ab}$ . The inverse of Eq. (1.2) is

$$e_a^\mu(x) = g^{\mu\nu} e_\nu^b \eta_{ba}. \quad (1.3)$$

From any set of space-time independent Dirac matrices satisfying

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}I,$$

one can solve Eq. (1.1) by choosing the space-time dependent Dirac matrices as particular linear combinations of the constant Dirac matrices:

$$\gamma^\mu(x) = e_a^\mu(x) \gamma^a. \quad (1.4)$$

This is not the most general solution to Eq. (1.1) but it does work. The action for the fermion fields and the Dirac equation is expressed in terms of these vierbeins [2–5]. The vierbein method is used for both the classical Dirac field [6–9] and for the quantized Dirac field [10,11].

There are several unattractive features of the vierbein formulation. (1) The 10 independent components of the metric tensor are replaced by the 16 components of the vierbeins. (2) It is necessary to introduce a special inertial frame at each point contrary to the basic principles that led Einstein to construct general relativity. Neither the inertial frames nor the constant Minkowski metric is necessary for spin 0 or spin 1 fields or for gravity itself. (3) When the vierbein solution for the Dirac matrices in Eq. (1.4) is inserted into the formula

$$\gamma_5 = -i \frac{\sqrt{-g}}{4!} \epsilon_{\alpha\beta\mu\nu} \gamma^\alpha \gamma^\beta \gamma^\mu \gamma^\nu, \quad (1.5)$$

all the space-time dependence cancels and  $\gamma_5$  becomes equal to the constant matrix  $\gamma_{(5)}$  from Minkowski space. Because of intrinsic parity violation, the matrix  $\gamma_5$  plays a central role in the electroweak interactions. The coupling of quarks and leptons to the weak vector bosons  $W^\pm$  and  $Z^0$  is through a linear combination of vector currents  $\bar{\psi} \gamma^\mu \psi$  and axial vector currents  $\bar{\psi} \gamma^\mu \gamma_5 \psi$ . In the vierbein formalism, local Lorentz transformations change the space-time dependence of the  $\gamma^\mu$ , but  $\gamma_5$  remains the same constant matrix as in Minkowski space.

In the standard vierbein formulation of fermion fields there are two types of transformations: general coordinate transformations and local Lorentz transformations. The Lagrangian density must be a scalar under both types of transformations.

*General Coordinate Transformations.* Under a general transformation  $x^\mu \rightarrow \tilde{x}^\mu(x)$  of the coordinate system, each of the four vierbeins transforms as a coordinate vector

$$\tilde{e}_a^\mu(\tilde{x}) = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} e_a^\mu(x). \quad (1.6)$$

The Dirac matrices transform as a vector:

$$\tilde{\gamma}^\mu(\tilde{x}) = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} \gamma^\nu(x). \quad (1.7a)$$

The transformations matrices  $\partial \tilde{x}^\mu / \partial x^\nu$  are  $4 \times 4$  real matrices belonging to the group  $\text{Gl}(4, \mathbb{R})$ . Because  $\text{Gl}(4, \mathbb{R})$  does not contain as a subgroup the spinor representations of the Lorentz group [10,12], it is not possible for the spinor field  $\psi(x)$  to transform under general coordinate transformations. Therefore the spinor field transforms as a scalar under general coordinate transformations:

$$\tilde{\psi}(\tilde{x}) = \psi(x). \quad (1.7b)$$

This is completely standard [2–5].

*Local Lorentz Transformations.* Conventionally the way to introduce spin base transformations that act on the components of the spinor quantities is to mimic Minkowski space. Under local Lorentz transformations of the inertial coordinate  $\xi^a \rightarrow \Lambda^a_b \xi^b$  the vierbeins mix according to

$$e_a^\mu(x) = e_b^\mu(x) \Lambda^b_a.$$

The constant Dirac matrices  $\gamma^a$  transform according to spinor representations of the local Lorentz group:

$$\Lambda^b_a \gamma^a = S_{\text{Lor}}(\Lambda) \gamma^b S_{\text{Lor}}(\Lambda).$$

Therefore under local Lorentz transformations of the inertial coordinate, the Dirac matrices and spinor field transform as

$$\gamma'^\mu(x) = S_{\text{Lor}}(\Lambda) \gamma^\mu(x) S_{\text{Lor}}^{-1}(\Lambda) \quad (1.8a)$$

$$\psi'(x) = S_{\text{Lor}}(\Lambda) \psi(x). \quad (1.8b)$$

The vierbein does not appear explicitly in the transformation equations. One may summarize this approach by saying that there are two types of transformations and the action must be a scalar under both. Coordinate transformations employ real  $4 \times 4$  matrices  $\partial \tilde{x}^\mu / \partial x^\nu$ . The spin base transformations employ complex  $4 \times 4$  matrices  $S_{\text{Lor}}(\Lambda)$  that are spinor representations of the 6-parameter Lorentz group.

Given the necessity of separate laws for general coordinate transformations and for spin base transformations, it seems natural to allow any complex  $4 \times 4$  matrix as a spin base transformation. To do this requires abandoning the vierbeins and dispensing with the privileged position of local inertial frames and local Lorentz transformations.

The remainder of the paper formulates the Dirac spinor field in a Riemannian space without using vierbeins. Section II develops two essential ingredients: the spin metric  $h$  which allows  $\psi^\dagger h \psi$  to be invariant under spin base transformations and the spin connection  $\Gamma_\mu$  which makes  $(\partial_\mu + \Gamma_\mu) \psi$  transform covariantly under a general spin base transformation [13]. Section III treats the equations of motion for the fermion field. Requiring the action to be stationary under variations in  $\psi$  gives the Dirac equation. Requiring it to be stationary under variations of  $g^{\mu\nu}$  gives the Einstein field equation containing the energy-momentum tensor of the fermions. Section IV provides a summary. There are five appendices. Appendix A summarizes the Clifford algebra basis. Appendix B proves a general theorem which is employed in the development of the spin connection, of the spin metric, and of the energy-momentum tensor. Appendix C shows which components of the general spin connection actually appear in the action for the fermion. Appendix D develops the second order Dirac equation and relates the spin curvature tensor to the Riemann-Christoffel event curvature tensor. Appendix E shows that with any fixed set of Dirac matrices satisfying Eq. (1.1) how to construct a spin base transformation to change the Dirac matrices to a vierbein basis.

## II. BASIC STRUCTURE

In the following it is not necessary to have an explicit form for the Dirac matrices which satisfy the basic anticommutation relation, Eq. (1.1).

### A. Spin metric

The matrices  $\gamma^{\mu\dagger}$ , which result from taking the transpose and the complex conjugate of  $\gamma^\mu$ , automatically satisfy the same anticommutation relation. As proven in Appendix B, the two solutions must be related by a matrix transformation:

$$\gamma^{\mu\dagger} = h \gamma^\mu h^{-1}, \quad (2.1)$$

for some matrix  $h$ . The adjoint of this relation can be used to obtain

$$[h^{-1} h^\dagger, \gamma^\mu] = 0$$

A basis for the 15 nontrivial matrices of the Clifford algebra can be formed from products of the  $\gamma^\mu$ . Since all of these will commute with  $h^{-1} h^\dagger$ , the latter must be proportional to the identity matrix:  $h^{-1} h^\dagger = zI$  for some complex number  $z$ . Taking the determinant of both sides shows that  $|z| = 1$  and that  $(\text{Det } h)^2 = z^{-4}$ . It is convenient to redefine  $h$  by changing  $h \rightarrow h\sqrt{z}$ . Then  $h^{-1} h^\dagger = \sqrt{z z^*} = 1$  so that  $h$  is hermitian:

$$h^\dagger = h, \quad (2.2)$$

and  $\text{Det } h = 1$ . The matrix  $h$  will be the spin metric. The definition in Eq. (2.1) also implies that

$$\gamma_5^\dagger = -h\gamma_5 h^{-1}$$

It is helpful to note that the matrices  $h\gamma^\mu$  and  $h\gamma^\mu\gamma_5$  are hermitian, whereas  $h\gamma_5$  and  $h[\gamma^\mu, \gamma^\nu]$  are anti-hermitian.

## B. Spin connection

Covariant derivatives of tensor quantities require the Riemann-Christoffel event connection

$$\Gamma_{\mu\lambda}^\nu = \frac{1}{2}g^{\nu\alpha}(\partial_\mu g_{\alpha\lambda} + \partial_\lambda g_{\alpha\mu} - \partial_\alpha g_{\mu\lambda}). \quad (2.3)$$

Covariant derivatives of spinors require the spin connection. Whatever choice is made for the Dirac matrices  $\gamma^\nu$ , one can compute the derivatives  $\partial_\mu \gamma^\nu$ . From these derivatives one can compute the coefficients

$$t_\mu^{\alpha\beta} = \frac{-1}{32}\text{Tr}\left(\gamma^\alpha(\partial_\mu \gamma^\beta + \Gamma_{\mu\lambda}^\beta \gamma^\lambda)\right) \quad (2.4a)$$

$$v_\mu^\alpha = \frac{1}{48}\text{Tr}\left([\gamma^\alpha, \gamma_\nu] \partial_\mu \gamma^\nu\right) \quad (2.4b)$$

$$a_\mu^\alpha = \frac{1}{8}\text{Tr}(\gamma_5 \partial_\mu \gamma^\alpha) \quad (2.4c)$$

$$p_\mu = \frac{1}{32}\text{Tr}(\gamma_5 \gamma_\nu \partial_\mu \gamma^\nu). \quad (2.4d)$$

It is easy to check the antisymmetry  $t_\mu^{\alpha\beta} + t_\mu^{\beta\alpha} = 0$ . For a fixed choice of  $\mu$  there are 15 complex coefficients: 6 independent values of  $t_\mu^{\alpha\beta}$ , 4 values of  $v_\mu^\alpha$ , 4 values of  $a_\mu^\alpha$ , and one  $p_\mu$ . Under general coordinate transformations each of these transforms as a tensor as indicated by the indices. (For comparison, with the vierbein solution for the Dirac matrices the coefficients  $v_\mu^\alpha$ ,  $a_\mu^\alpha$ ,  $p_\mu$ , and  $s_\mu$  all vanish; and the nonvanishing term  $t_\mu^{\alpha\beta}$  has 6 real coefficients.)

The spin connection is  $\Gamma_\mu$  has the general decomposition in terms Dirac matrices as

$$\Gamma_\mu = t_\mu^{\alpha\beta}[\gamma_\alpha, \gamma_\beta] + v_\mu^\alpha \gamma_\alpha + a_\mu^\alpha \gamma_\alpha \gamma_5 + p_\mu \gamma_5 + s_\mu I, \quad (2.5)$$

where  $s_\mu$  is undetermined. The letters  $t, v, a, p, s$  denote tensor, vector, axial vector, pseudoscalar, and scalar. Appendix B shows that the spin connection given in Eq. (2.5) satisfies

$$\partial_\mu \gamma^\nu + \Gamma_{\mu\lambda}^\nu \gamma^\lambda + [\Gamma_\mu, \gamma^\nu] = 0. \quad (2.6)$$

It is elementary that if  $\Gamma_\mu$  is postulated to satisfy Eq. (2.6) then the coefficients are as given in Eq. (2.4). Appendix B proves the converse, that Eq. (2.6) is actually satisfied. The result is non-trivial and it is worth doing some counting to appreciate what has happened. For a fixed choice of  $\mu$  and  $\nu$  the derivative  $\partial_\mu \gamma^\nu$  will generally be a linear combination of 15 Dirac matrices. If only  $\mu$  is

fixed and  $\nu$  runs over its four values then Eq. (2.6) is a set of 60 linear equations. These 60 equations are solved by the 15 coefficients displayed in Eq. (2.4).

Eq. (2.6) also determines the derivative of products of Dirac matrices. In particular, it gives

$$\partial_\mu \gamma_5 + [\Gamma_\mu, \gamma_5] = 0.$$

## C. Spinor fields

The covariant derivative of the fermion field  $\psi$  is

$$\nabla_\mu \psi = \partial_\mu \psi + \Gamma_\mu \psi. \quad (2.7)$$

The fermion action is  $A_f = \int d^4x \sqrt{-g} \mathcal{L}_f$ , where the Lagrangian density is

$$\mathcal{L}_f = \frac{i}{2}\psi^\dagger h\gamma^\mu \nabla_\mu \psi - \frac{i}{2}(\nabla_\mu \psi)^\dagger h\gamma^\mu \psi - m\psi^\dagger h\psi. \quad (2.8)$$

To examine the role of the spin connection it is helpful to write out  $\mathcal{L}$  in detail:

$$\begin{aligned} \mathcal{L} = & \frac{i}{2}\psi^\dagger h\gamma^\mu \partial_\mu \psi - \frac{i}{2}(\partial_\mu \psi)^\dagger h\gamma^\mu \psi - m\psi^\dagger h\psi \\ & + \frac{i}{2}\psi^\dagger (h\gamma^\mu \Gamma_\mu - \Gamma_\mu^\dagger h\gamma^\mu) \psi. \end{aligned}$$

When the full spin connection is substituted in the action there are numerous cancellations so that the 16 complex parameters are reduced to 16 real parameters. (See Appendix C.) Of these cancellations, the simplest occurs in the the part of  $\Gamma_\mu$  that is proportional to the identity matrix, viz.  $s_\mu I$ . In particular  $(\text{Re } s_\mu)I$  does not appear in the action. It is very convenient to subtract off  $2(\text{Re } s_\mu)I$  from the spin connection and define

$$\hat{\Gamma}_\mu = \Gamma_\mu - \frac{1}{4}\text{Re}[\text{Tr}(\Gamma_\mu)]I. \quad (2.9a)$$

The subtracted spin connection satisfies

$$\text{Re}[\text{Tr}(\hat{\Gamma}_\mu)] = 0. \quad (2.9b)$$

It will be convenient to use  $\hat{\Gamma}_\mu$  in Sec. III for the discussion of the Dirac equation and the energy-momentum tensor.

## D. Spin base transformations

Starting from a set of Dirac matrices  $\gamma^\mu$  satisfying the anticommutation relation Eq. (1.1) one can change to a new set by a spin base transformation

$$\gamma'^\nu = S\gamma^\nu S^{-1}, \quad (2.10)$$

where  $S$  is an invertible, complex  $4 \times 4$  matrix with arbitrary dependence on space-time. Thus  $S$  belongs to the

32 parameter group  $\text{Gl}(4, \mathbb{C})$ . Under a spin base transformation the derivative of the Dirac matrices transforms inhomogeneously:

$$\partial_\mu \gamma'^\nu = S \left( \partial_\mu \gamma^\nu + [S^{-1} \partial_\mu S, \gamma^\nu] \right) S^{-1}$$

The coefficients in Eq. (2.4) also change inhomogeneously and this gives the transformed spin connection

$$\Gamma'_\mu = S \Gamma_\mu S^{-1} - S^{-1} \partial_\mu S. \quad (2.11)$$

The transformed Dirac matrix and the transformed spin connection satisfy

$$\partial_\mu \gamma'^\nu + \Gamma'_{\mu\lambda} \gamma'^\lambda + [\Gamma'_\mu, \gamma'^\nu] = 0. \quad (2.12)$$

The event connection  $\Gamma'_{\mu\lambda}$  does not change under a spin base transformation.

Under a spin base transformation the spinor field behaves as

$$\psi' = S \psi. \quad (2.13a)$$

The covariant derivative defined in Eq. (2.7) transforms homogeneously:

$$\nabla'_\mu \psi' = S \nabla_\mu \psi. \quad (2.13b)$$

The adjoints of the new Dirac matrices satisfy

$$\gamma'^{\mu\dagger} = h' \gamma'^\mu h'^{-1}, \quad (2.14a)$$

where the transformed spin metric is

$$h' = S^{\dagger-1} h S^{-1}. \quad (2.14b)$$

There are several comparisons to be made. (1) The fact that the spin metric changes under a spin base transformation is analogous to the fact that the event metric  $g_{\mu\nu}$  changes as a result of coordinate transformations. (2) By such a transformation it is always possible to make  $h'$  a constant matrix with a convenient form, e.g. diagonal. However subsequent spin base transformations will change  $h'$ , unless one arbitrarily limits the allowed spin base transformations. (3) One can rewrite the above relation as

$$S^\dagger = h S^{-1} h'^{-1}.$$

This form emphasizes the relation of  $S^\dagger$  to  $S^{-1}$ . If one artificially limits the the allowed spin base transformations to those which do not change the spin metric, then  $h' = h$  so that  $S^\dagger$  is matrix equivalent to  $S^{-1}$ . Such a choice would limit  $S$  to be in the 16-parameter, unitary group  $\text{U}(2,2)$  [14]. In what follows this restriction will not be made:  $h$  and  $h'$  will not be constant and the spin base transformations will not be required to keep the spin metric invariant.

Associated with each spinor field  $\psi$  is the Dirac adjoint field

$$\bar{\psi} = \psi^\dagger h \quad (2.15)$$

The product  $\bar{\psi}\psi$  is manifestly self-adjoint. Under a spin base transformation the Dirac adjoint field  $\bar{\psi}$  changes into

$$\bar{\psi}' = \psi'^\dagger h' = \psi^\dagger h S^{-1} = \bar{\psi} S^{-1} \quad (2.16)$$

This provides various invariants under spin base transformations:

$$\begin{aligned} \bar{\psi}' \psi' &= \bar{\psi} \psi \\ \bar{\psi}' \gamma'_\nu \nabla'_\mu \psi' &= \bar{\psi} \gamma_\nu \nabla_\mu \psi \\ \bar{\psi}' \gamma'_\nu \psi' &= \bar{\psi} \gamma_\nu \psi \end{aligned}$$

The first of these is the mass term in the fermion action; the second is part of the energy-momentum tensor; and the third is the vector current for electromagnetism.

## E. Additional covariant derivatives

The covariant derivatives of various additional quantities will be needed. Since  $\bar{\psi}\psi$  is a coordinate scalar and a spin base scalar, the Leibnitz product rule gives  $\partial_\mu(\bar{\psi}\psi) = (\nabla_\mu \bar{\psi})\psi + \bar{\psi}(\nabla_\mu \psi)$ . Consequently the covariant derivative of the Dirac adjoint field is

$$\nabla_\mu \bar{\psi} = \partial_\mu \bar{\psi} - \bar{\psi} \Gamma_\mu. \quad (2.17)$$

On the other hand, the adjoint of Eq. (2.7) is

$$\nabla_\mu \psi^\dagger = \partial_\mu \psi^\dagger + \psi^\dagger \Gamma_\mu^\dagger \quad (2.18)$$

where  $\Gamma_\mu^\dagger$  is the complex conjugate, transpose matrix. Comparing these last two results gives the covariant derivative of the spin metric:

$$\nabla_\mu h = \partial_\mu h - h \Gamma_\mu - \Gamma_\mu^\dagger h. \quad (2.19)$$

It is possible to evaluate this. Taking the complex conjugate, transpose of Eq. (2.6) and using the definition of the spin metric leads to

$$0 = \left[ h^{-1} \partial_\mu h - \Gamma_\mu - h^{-1} \Gamma_\mu^\dagger h, \gamma^\nu \right].$$

By Schur's lemma the only matrix than commutes with all the  $\gamma^\nu$  is a multiple of the identity matrix so that

$$h^{-1} \partial_\mu h - \Gamma_\mu - h^{-1} \Gamma_\mu^\dagger h = c_\mu I$$

One can evaluate  $c_\mu$  by taking the the trace of both sides of this relation:

$$c_\mu = -\frac{1}{2} \text{Re} [\text{Tr}(\Gamma_\mu)].$$

In terms of the subtracted spin connection  $\hat{\Gamma}_\mu$  defined in Eq. (2.9a) this reads

$$h^{-1}\partial_\mu h - \widehat{\Gamma}_\mu - h^{-1}\widehat{\Gamma}_\mu^\dagger h = 0. \quad (2.20a)$$

This relation is also useful as a way of computing the adjoint of the spin connection:

$$\widehat{\Gamma}_\mu^\dagger = -h\widehat{\Gamma}_\mu h^{-1} + (\partial_\mu h)h^{-1}. \quad (2.20b)$$

A third possibility is to view Eq. (2.20a) as the vanishing of the covariant derivative of the spin metric:

$$\widehat{\nabla}_\mu h = \partial_\mu h - h\widehat{\Gamma}_\mu - \widehat{\Gamma}_\mu^\dagger h = 0. \quad (2.20c)$$

### III. FIELD EQUATIONS

The action for the fermion field is  $A_f = \int d^4x \sqrt{-g} \mathcal{L}_f$ , where the Lagrangian density for fermions is

$$\mathcal{L}_f = \frac{i}{2}\psi^\dagger h\gamma^\mu \nabla_\mu \psi - \frac{i}{2}(\nabla_\mu \psi)^\dagger h\gamma^\mu \psi - m\bar{\psi}\psi. \quad (3.1)$$

The independent variables are the fermion field  $\psi$ , its adjoint  $\psi^\dagger$  and the metric tensor  $g^{\mu\nu}$ . The Dirac matrices, the spin connection, and the spin metric depend on the event metric  $g_{\mu\nu}$ . There is no vierbein. The Lagrangian density is a scalar under general coordinate transformations and invariant under general spin base transformations in  $\text{Gl}(4, \mathbb{C})$ .

#### A. Dirac equation

As discussed in Sec. II C, the real part of the trace of the spin connection  $\Gamma_\mu$  does not contribute to the action and it is convenient to discard that part by employing  $\widehat{\Gamma}_\mu$  as defined in Eq. (2.9a) and define the matrix differential operator

$$K = ih\gamma^\mu(\partial_\mu + \widehat{\Gamma}_\mu) - mh. \quad (3.2)$$

The fermion action can be written

$$A_f = \int d^4x \sqrt{-g} \left\{ \frac{1}{2}\psi^\dagger K\psi + \frac{1}{2}(K\psi)^\dagger \psi \right\}, \quad (3.3)$$

By using the identity

$$\partial_\mu(\sqrt{-g} h\gamma^\mu) = \sqrt{-g}(h\gamma^\mu \widehat{\Gamma}_\mu + \widehat{\Gamma}_\mu^\dagger h\gamma^\mu),$$

it is easy to show that for any two spinor fields  $\psi$  and  $\chi$  that fall-off sufficiently fast ,

$$\int d^4x \sqrt{-g} \chi^\dagger K\psi = \int d^4x \sqrt{-g} (K\chi)^\dagger \psi. \quad (3.4)$$

Thus  $\sqrt{-g} K$  is a self-adjoint operator.

Extremizing the action with respect to  $\psi^\dagger$  gives the generalized Dirac equation  $K\psi = 0$ , or more explicitly

$$i\gamma^\mu(\partial_\mu + \widehat{\Gamma}_\mu)\psi = m\psi. \quad (3.5a)$$

Varying the action with respect to  $\psi$  gives

$$-i(\partial_\mu \bar{\psi})\gamma^\mu + i\bar{\psi}\widehat{\Gamma}_\mu\gamma^\mu = m\bar{\psi}. \quad (3.5b)$$

The second equation is also implied by the first. As a consequence of these the vector current and the axial vector current obey the following:

$$\begin{aligned} \partial_\mu(\bar{\psi}\gamma^\mu\psi) + \Gamma_{\mu\lambda}^\mu \bar{\psi}\gamma^\lambda\psi &= 0 \\ \partial_\mu(\bar{\psi}\gamma^\mu\gamma_5\psi) + \Gamma_{\mu\lambda}^\mu \bar{\psi}\gamma^\lambda\gamma_5\psi &= i2m\bar{\psi}\psi, \end{aligned}$$

with the axial anomaly omitted. Appendix D iterates the Dirac equation (3.5a) to obtain the second order form. The anticommutator of the covariant derivatives  $[\nabla_\mu, \nabla_\nu]\psi$  introduces the spin curvature tensor, which is directly related to the Riemann-Christoffel curvature tensor, so as to simplify the second order wave equation.

#### B. Energy-momentum tensor

The energy-momentum tensor, being the source of the gravitational field, must also be computed. The fermion contribution to the energy-momentum tensor requires varying the fermion action with respect to general changes in the metric tensor:

$$\delta S = \frac{1}{2} \int d^4x \sqrt{-g} (\delta g^{\mu\nu}) T_{\mu\nu}.$$

1. The theorem proven in Appendix B provides the variational derivative of the Dirac matrices, of the spin metric, and of the spin connection. The most general change in the Dirac matrices that can result from a change in the metric tensor is

$$\delta\gamma^\mu = \frac{1}{2}(\delta g^{\mu\nu})\gamma_\nu + [\gamma^\mu, G], \quad (3.6a)$$

where  $G$  is some  $4 \times 4$  matrix. Without loss of generality one may restrict  $G$  to be traceless since the identity matrix commutes with  $\gamma^\mu$ . The dependence of the matrix  $G$  on the metric tensor will depend upon how the basic anticommutator in Eq. (1.1) is solved. (Even for the vierbein solution  $G$  is not zero.)

2. To compute the variation in the spin metric  $h$ , take the variation of Eq. (2.1) and substitute Eq. (3.6a) to get

$$0 = [h^{-1}\delta h - G - h^{-1}G^\dagger h, \gamma^\mu].$$

By Schur's lemma the only matrix that commutes with all the  $\gamma^\mu$  is a multiple of the identity. Since  $\text{Tr}(h^{-1}\delta h) = 0$  and  $\text{Tr}(G) = 0$ , the quantity that commutes is traceless, which implies

$$\delta h = hG + G^\dagger h. \quad (3.6b)$$

The product  $h\gamma^\mu$  appears throughout the action. Its variation is therefore

$$\delta(h\gamma^\mu) = \frac{1}{2}(\delta g^{\mu\nu})h\gamma_\nu + h\gamma^\mu G + G^\dagger h\gamma^\mu.$$

3. Next one needs the dependence of the spin connection  $\Gamma_\mu$  on the metric. Varying Eq. (2.6) with respect to the metric tensor gives

$$0 = \nabla_\mu(\delta\gamma^\nu) + (\delta\Gamma_{\mu\lambda}^\nu)\gamma^\lambda + [\delta\Gamma_\mu, \gamma^\nu].$$

The first term can be evaluated using Eq. (3.6a) and the fact that  $\nabla_\mu\gamma^\lambda = 0$ :

$$\nabla_\mu(\delta\gamma^\nu) = \frac{1}{2}(\nabla_\mu\delta g^{\nu\lambda})\gamma_\lambda - [\nabla_\mu G, \gamma^\nu].$$

By taking the variation of  $0 = \nabla_\mu g^{\nu\lambda}$  this can be expressed more explicitly as

$$\nabla_\mu(\delta\gamma^\nu) = -\frac{1}{2}(\delta\Gamma_{\mu\lambda}^\nu)\gamma^\lambda - \frac{1}{2}(\delta\Gamma_{\mu\alpha}^\lambda)g^{\alpha\nu}\gamma_\lambda - [\nabla_\mu G, \gamma^\nu].$$

Substituting above gives

$$0 = \frac{1}{2}(\delta\Gamma_{\mu\lambda}^\nu)\gamma^\lambda - \frac{1}{2}(\delta\Gamma_{\mu\alpha}^\lambda)g^{\alpha\nu}\gamma_\lambda + [\Gamma_\mu - \nabla_\mu G, \gamma^\nu].$$

The first two terms together can be written as a commutator with  $\gamma^\nu$  so that

$$0 = [\Gamma_\mu - \nabla_\mu G - \frac{1}{8}\delta\Gamma_{\mu\beta}^\alpha [\gamma_\alpha, \gamma^\beta], \gamma^\nu].$$

Since each term on the left hand side of the commutator is traceless, Schur's lemma implies that

$$\delta\Gamma_\mu = \partial_\mu G + [\Gamma_\mu, G] + \frac{1}{8}(\delta\Gamma_{\mu\beta}^\alpha) [\gamma_\alpha, \gamma^\beta]. \quad (3.6c)$$

4. To compute the energy-momentum tensor for a fermion field requires varying the action given in Eq. (3.1):

$$\begin{aligned} \delta S = & \int d^4x (\delta\sqrt{-g}) \left\{ \frac{1}{2}\psi^\dagger K\psi + \frac{1}{2}(K\psi)^\dagger\psi \right\} \\ & + \int d^4x \sqrt{-g} \left\{ \frac{1}{2}\psi^\dagger(\delta K)\psi + \frac{1}{2}((\delta K)\psi)^\dagger\psi \right\} \end{aligned}$$

The variation of the differential operator  $K$  given in Eq. (3.2) with respect to the metric tensor can be computed using Eqs. (3.6a), (3.6b), and (3.6c) with the result:

$$\begin{aligned} \delta K = & \frac{i}{2}(\delta g^{\mu\nu})h\gamma_\nu(\partial_\mu + \hat{\Gamma}_\mu) + KG + G^\dagger K \\ & + \frac{i}{8}(\delta\Gamma_{\mu\beta}^\alpha)h\gamma^\mu[\gamma_\alpha, \gamma^\beta] \end{aligned} \quad (3.7)$$

The Dirac equation,  $K\psi = 0$ , and the self-adjoint property Eq. (3.4) guarantees that the terms  $KG$  and  $G^\dagger K$  will make no contribution. Consequently it was never necessary to know the matrix  $G$  explicitly. The last term

in Eq. (3.7) is anti-hermitian and automatically cancels in  $\delta S$ . Thus the variation gives

$$\delta S = \int d^4x \sqrt{-g} \delta g^{\mu\nu} \left\{ \frac{i}{4}\psi^\dagger h\gamma_\nu \nabla_\mu \psi - \frac{i}{4}(\nabla_\mu \psi)^\dagger h\gamma_\nu \psi \right\}$$

Symmetrizing with respect to  $\mu\nu$  gives the final result for the energy-momentum tensor for any fermion field :

$$T_{\mu\nu} = \frac{i}{4}(\bar{\psi}\gamma_\nu \nabla_\mu \psi + \bar{\psi}\gamma_\mu \nabla_\nu \psi) \quad (3.8)$$

$$- \frac{i}{4}((\nabla_\mu \bar{\psi})\gamma_\nu \psi + (\nabla_\nu \bar{\psi})\gamma_\mu \psi). \quad (3.9)$$

The Einstein field equations are, as always,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}. \quad (3.10)$$

## IV. SUMMARY AND CONCLUSIONS

The action for fermions that has been constructed is invariant under two separate transformations: general coordinate transformations and local spin base transformations.

1. *General Coordinate Transformations.* Under a general transformation  $x^\mu \rightarrow \tilde{x}^\mu(x)$  of the coordinate system the Dirac matrices and the fermion field transform as

$$\tilde{\gamma}^\mu(\tilde{x}) = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} \gamma^\nu(x) \quad (4.1a)$$

$$\tilde{\psi}(\tilde{x}) = \psi(x). \quad (4.1b)$$

This is completely standard.

2. *Spin Base Transformations.* Under arbitrary spin base transformations the Dirac matrices and spinor field transform as

$$\gamma'^\mu(x) = S(x)\gamma^\mu(x)S^{-1}(x) \quad (4.2a)$$

$$\psi'(x) = S(x)\psi(x), \quad (4.2b)$$

where  $S(x)$  is a any matrix in  $Gl(4, c)$ .

3. *Spin base transformations to vierbein base.* According to the theorem proven in Appendix B, any two sets of Dirac matrices that solve the anticommutation relations must be related by a spin-base similarity transformation. Consequently every solution to the anticommutation relations is spin-base equivalent to a vierbein solution. Appendix E shows how, starting from an arbitrary set of Dirac matrices  $\gamma^\mu$ , to construct a transformation matrix  $S$  and the quantities  $E_a^\mu$  satisfying

$$\gamma^\mu = E_a^\mu S\gamma^a S^{-1}. \quad (4.3)$$

The construction does not prove that  $E_a^\mu$  are derivatives of a locally inertial coordinate as are the vierbeins in Eq. (1.2).

4. *Inclusion of gauge fields.* Every species of fermion field contains the covariant derivative  $(\partial_\mu + \hat{\Gamma}_\mu)\psi$ . Under a spin base transformation every fermion field undergoes the same transformation  $\psi \rightarrow S\psi$ . By contrast, nonabelian gauge transformations always transform each fermion in a multiplet differently. For example, in QCD each type of quark transforms as a triplet of the SU(3) color group. In the electroweak interactions the chiral projections  $\frac{1}{2}(1 - \gamma_5)\psi$  of the quarks and leptons transform as doublets of the gauge group SU(2) and these gauge transformations are distinct from spin base transformations. However the full electroweak gauge group is SU(2)×U(1) and the abelian U(1) factor requires some discussion.

It is simplest consider QED and then return to the electroweak U(1) gauge invariance. Let  $\psi_e$  be the field of the electron field;  $\psi_u$ , the field of the up quark; and  $\psi_d$ , the field of the down quark. The kinetic terms in the action are

$$\begin{aligned} \bar{\psi}_e(\partial_\mu + \hat{\Gamma}_\mu - ieA_\mu)\psi_e + \bar{\psi}_u(\partial_\mu + \hat{\Gamma}_\mu + i\frac{2}{3}eA_\mu)\psi_u \\ + \bar{\psi}_d(\partial_\mu + \hat{\Gamma}_\mu - i\frac{1}{3}eA_\mu)\psi_d. \end{aligned}$$

Included among the spin base transformations is the phase change  $\psi \rightarrow \exp(-i\phi)\psi$  of all three fields, where  $\phi$  is an arbitrary real function. Under this spin base transformation  $\hat{\Gamma}_\mu \rightarrow \hat{\Gamma}_\mu + i(\partial_\mu\phi)I$  and  $A_\mu$  does not change. By contrast, under an electromagnetic gauge transformation each fermion field transforms with a different phase:  $\psi_e \rightarrow \exp(-i\phi)\psi_e$ ,  $\psi_u \rightarrow \exp(i\frac{2}{3}\phi)\psi_u$ , and  $\psi_d \rightarrow \exp(-i\frac{1}{3}\phi)\psi_d$ . The vector potential changes,  $A_\mu \rightarrow A_\mu - \partial_\mu\phi/e$ , but  $\hat{\Gamma}_\mu$  does not change.

The behavior of the abelian group in the SU(2)×U(1) electroweak interactions is analogous except that there is parity violation in the coupling to the U(1) vector potential  $B_\mu$ . Included among the spin base transformations  $\psi \rightarrow S\psi$  are those for which  $S^{-1}\partial_\mu S = i\partial_\mu\phi(aI + b\gamma_5)$  with  $a$  and  $b$  the same for all fermions. The U(1) gauge transformation is distinct from these spin base transformations in that the values of  $a$  and  $b$  are different for each field type and consequently the vector potential  $B_\mu$  changes but the spin connection does not.

5. *Non-Riemannian spaces.* This paper treats only Riemann spaces. In a non-Riemannian space there additional degrees of freedom beyond the metric that determine the geometry. The full event connection is the sum of the Christoffel connection and an additional event connection. The full spin connection is the sum of the Riemann spin connection used here and an additional piece representing the new degrees of freedom. There are no obstacles encountered in extending the above treatment to this more general space without using vierbeins. The full Gl(4,c) invariance is maintained.

## ACKNOWLEDGMENTS

It is a pleasure to thank Richard Treat for many instructive discussions regarding this paper. This work was supported in part by National Science Foundation grant PHY-9900609.

## APPENDIX A: CLIFFORD ALGEBRA

In the anticommutation relations Eq. (1.1) the off-diagonal components of the general metric  $g^{\mu\nu}$  mean that the anticommutator is never zero. In proving various results it is often much easier to deal with one covariant index and one contravariant index so that

$$\{\gamma_\mu, \gamma^\nu\} = 2\delta_\mu^\nu I.$$

Using this one can show that the space-time dependent matrix  $\gamma_5$  defined in Eq. (1.5) has the property

$$\gamma_5 \gamma^\nu \gamma_5 = -\gamma^\nu. \quad (A1)$$

Contracting both sides with  $\gamma_\nu$  gives

$$\gamma_5 \gamma_5 = I. \quad (A2)$$

Taking trace of Eq.(A1) and using the cyclic property gives  $\text{Tr}(\gamma^\nu) = -\text{Tr}(\gamma^\nu)$  and therefore

$$\text{Tr}(\gamma^\nu) = 0. \quad (A3)$$

Because  $\gamma_\nu \gamma^\nu = 4$ , Eq. (A1) implies  $\gamma_5 = -\gamma_\nu \gamma_5 \gamma^\nu / 4$ . Taking the trace of this and using the cyclic property gives  $\text{Tr}(\gamma_5) = -\text{Tr}(\gamma_5)$  and therefore

$$\text{Tr}(\gamma_5) = 0. \quad (A4)$$

The four Dirac matrices are the bases for the Clifford algebra. The vector space of this algebra is spanned by 16 matrices which can be chosen as the covariant tensors 1,  $\gamma_\alpha$ ,  $[\gamma_\alpha, \gamma_\beta]$ ,  $\gamma_5$ , and  $\gamma_\alpha \gamma_5$ . All except the identity are traceless.

## APPENDIX B: GENERAL THEOREM

There are several computations which require knowing how the Dirac matrices change when the metric tensor changes in a specified way. The answer to this comes from the fundamental anticommutation relation

$$\{\gamma^\nu, \gamma^\kappa\} = 2g^{\nu\kappa} I. \quad (B1)$$

Under an infinitesimal change in the metric tensor this becomes

$$\{\Delta\gamma^\nu, \gamma^\kappa\} + \{\gamma^\nu, \Delta\gamma^\kappa\} = 2\Delta g^{\nu\kappa} I. \quad (B2)$$

This Appendix proves that the most general solution for  $\Delta\gamma^\nu$  is

$$\Delta\gamma^\nu = \frac{1}{2}(\Delta g^{\nu\lambda})\gamma_\lambda - [M, \gamma^\nu], \quad (\text{B3})$$

where  $M$  is a  $4 \times 4$  matrix. The specific values of the change  $\Delta g^{\nu\lambda}$  will determine the specific value of  $M$ . Before proving the theorem it may be helpful to see the actual uses of this theorem.

### 1. The spin connection

One application of the theorem is to compute the space-time derivative of the Dirac matrices in terms of the derivatives of the metric. In other words, express  $\Delta\gamma^\nu = dx^\mu \partial_\mu \gamma^\nu$  in terms of  $\Delta g^{\nu\lambda} = dx^\mu \partial_\mu g^{\nu\lambda}$ . In this case the matrix  $M$  must also be proportional to the coordinate differentials:  $M = dx^\mu M_\mu$ . Since  $\nabla_\mu g^{\nu\lambda} = 0$  it follows that

$$\Delta g^{\nu\lambda} = -dx^\mu (\Gamma_{\mu\kappa}^\nu g^{\kappa\lambda} + \Gamma_{\mu\kappa}^\lambda g^{\kappa\nu}).$$

Then Eq. (B3) can be rearranged as

$$dx^\mu (\partial_\mu \gamma^\nu + \Gamma_{\mu\lambda}^\nu \gamma^\lambda) = \frac{1}{2} dx^\mu (\Gamma_{\mu\kappa}^\nu g^{\kappa\lambda} - \Gamma_{\mu\kappa}^\lambda g^{\kappa\nu}) \gamma_\lambda - dx^\mu [M_\mu, \gamma^\nu].$$

Because the two terms on the right involving the event connection are antisymmetric under  $\nu \leftrightarrow \lambda$ , the entire right hand side can be written as commutator with  $\gamma^\nu$ :

$$\partial_\mu \gamma^\nu + \Gamma_{\mu\lambda}^\nu \gamma^\lambda = -[\Gamma_\mu, \gamma^\nu] \quad (\text{B4})$$

The matrix  $\Gamma_\mu$  is the spin connection:

$$\Gamma_\mu = \frac{1}{8} \Gamma_{\mu\beta}^\alpha [\gamma_\alpha, \gamma^\beta] + M_\mu. \quad (\text{B5})$$

Although  $M_\mu$  is not a vector under general coordinate transformations, the spin connection  $\Gamma_\mu$  is automatically a vector.

### 2. Pauli's theorem in curved space

Given one set of Dirac matrices  $\gamma^\nu$  satisfying Eq. (B1) and another set  $\gamma'^\nu$  which also satisfying Eq. (B1), it is natural to ask if the two solutions are related. In Minkowski space Pauli proved that the two sets are always related by a similarity transformation [16–18].

In curved space-time, this question is equivalent to asking what infinitesimal changes  $\Delta\gamma^\nu$  are possible when  $\Delta g^{\nu\kappa} = 0$ . The most general solution given in Eq. (B3) is that  $\gamma^\nu + \Delta\gamma^\nu$  is of the form  $\gamma^\nu - [M, \gamma^\nu]$  for infinitesimal  $M$ . Iterating this shows that the most general solution if of the form

$$\exp(-M) \gamma^\nu \exp(M). \quad (\text{B6})$$

Thus any two sets of Dirac matrices satisfying anticommutation relations (B1) with the same metric tensor, must be related by a similarity transformation. This is used in Sec. II C, where the spin metric  $h$  is the similarity transformation between  $\gamma^\nu$  and  $\gamma'^\nu$  and in Appendix E.

### 3. The energy-momentum tensor

Another application occurs in the computation of the energy-momentum tensor in Sec. III B. There the problem is to compute the change in the Dirac matrices produced by an arbitrary variation in the metric  $\Delta g^{\nu\kappa}$ .

### 4. Proof of the theorem

To prove Eq. (B3) the first step is to parameterize the most general possible change in the Dirac matrices as

$$\Delta\gamma^\nu = 8T^{\nu\alpha}\gamma_\alpha + 2A^\nu\gamma_5 + B^{\nu\alpha}\gamma_5\gamma_\alpha + C^{\nu\alpha\beta}[\gamma_\alpha, \gamma_\beta] \quad (\text{B7})$$

For a fixed value of  $\nu$  the expansion is a linear combination of 15 traceless matrices. As already noted in the above applications, the coefficients may or may not transform as coordinate tensors depending upon whether  $\Delta g^{\nu\lambda}$  is a tensor.

1. The coefficient  $T^{\nu\alpha}$  can be written as a trace:

$$T^{\nu\kappa} = \frac{1}{32} \text{Tr}((\Delta\gamma^\nu)\gamma^\kappa).$$

The symmetric part of this is

$$T^{\nu\kappa} + T^{\kappa\nu} = \frac{1}{8} \Delta g^{\nu\kappa} \quad (\text{B8})$$

2. Next substitute the expansion Eq. (B7) into Eq. (B2) to get

$$2\Delta g^{\nu\kappa} = 16(T^{\nu\kappa} + T^{\kappa\nu}) + \gamma_5 X^{\nu\kappa} + Y^{\nu\kappa},$$

where  $X$  and  $Y$  denote the matrices

$$X^{\nu\kappa} = B^{\nu\alpha}[\gamma_\alpha, \gamma^\kappa] + B^{\kappa\alpha}[\gamma_\alpha, \gamma^\nu] \\ Y^{\nu\kappa} = C^{\nu\alpha\beta}\{[\gamma_\alpha, \gamma_\beta], \gamma^\kappa\} + C^{\kappa\alpha\beta}\{[\gamma_\alpha, \gamma_\beta], \gamma^\nu\}.$$

Because of Eq. (B8) this becomes

$$0 = \gamma_5 X^{\nu\kappa} + Y^{\nu\kappa}.$$

From their definitions,  $\gamma_5$  commutes with  $X^{\nu\kappa}$  but anti-commutes with  $Y^{\nu\kappa}$ . Therefore both matrices vanish

$$X^{\nu\kappa} = 0 \quad Y^{\nu\kappa} = 0.$$



3. The vanishing of  $X$  allows us to extract information about the coefficients  $B^{\nu\alpha}$  by evaluating the commutator

$$0 = [X_{\mu}^{\nu\kappa}, \gamma_{\kappa}] = 16B^{\nu\alpha}\gamma_{\alpha} - 4(g_{\alpha\kappa}B^{\alpha\kappa})\gamma^{\nu}.$$

This fixes  $B^{\nu\alpha}$  to have the structure

$$B^{\nu\alpha} = -2P g^{\nu\alpha}, \quad (\text{B9})$$

where  $P$  is unknown.

4. The fact that  $Y = 0$  will yield a simplification in the  $C$  coefficients from which it is made. Define the anticommutator

$$Z^{\kappa} = \frac{1}{4}\{Y^{\nu\kappa}, \gamma_{\nu}\} = 0.$$

Explicit calculation gives

$$Z^{\kappa} = 3C^{\kappa\alpha\beta}[\gamma_{\alpha}, \gamma_{\beta}] + C^{\nu\alpha\beta}(g_{\nu\alpha}[\gamma_{\beta}, \gamma^{\kappa}] - g_{\nu\beta}[\gamma_{\alpha}, \gamma^{\kappa}]).$$

To eliminate the Dirac matrices, compute

$$\frac{1}{16}\text{Tr}(Z^{\kappa}[\gamma^{\lambda}, \gamma^{\rho}]) = 3C^{\kappa[\rho\lambda]} + g_{\nu\alpha}(C^{\nu[\alpha\rho]}g^{\kappa\lambda} - C^{\nu[\alpha\lambda]}g^{\kappa\rho}).$$

Since  $Z^{\kappa} = 0$ , the right hand side must vanish. Therefore the three-index coefficient has the structure

$$C^{\kappa\rho\lambda} - C^{\kappa\lambda\rho} = V^{\lambda}g^{\kappa\rho} - V^{\rho}g^{\kappa\lambda}. \quad (\text{B10})$$

with  $V^{\lambda}$  unknown.

5. Using the results of Eq. (B8), (B9), (B10) the expansion in Eq. (B7) simplifies to

$$\begin{aligned} \Delta\gamma^{\nu} = & \frac{1}{2}(\Delta g^{\nu\lambda})\gamma_{\lambda} + 4(T^{\nu\alpha} - T^{\alpha\nu})\gamma_{\alpha} + 2A^{\nu}\gamma_5 \\ & - 2P\gamma_5\gamma^{\nu} + V^{\beta}[\gamma^{\nu}, \gamma_{\beta}] \end{aligned} \quad (\text{B11})$$

Now define a matrix  $M$  by

$$M = T^{\alpha\beta}[\gamma_{\alpha}, \gamma_{\beta}] + A^{\alpha}\gamma_{\alpha}\gamma_5 + P\gamma_5 + V^{\alpha}\gamma_{\alpha} \quad (\text{B12})$$

Then Eq. (B11) can be written in the simple form

$$\Delta\gamma^{\nu} = \frac{1}{2}(\Delta g^{\nu\lambda})\gamma_{\lambda} - [M, \gamma^{\nu}]. \quad (\text{B13})$$

This proves the theorem quoted in Eq. (B3).

### APPENDIX C: CONTRIBUTION OF THE SPIN CONNECTION TO THE ACTION

As mentioned in Sec. II, the full spin connection is parameterized by 16 complex coefficients. Some parts of the spin connection automatically cancel out of the fermion action. The spin connection appears in the action through the combination

$$\frac{i}{2}\psi^{\dagger}\left[h\gamma^{\mu}\Gamma_{\mu} - \left(h\gamma^{\mu}\Gamma_{\mu}\right)^{\dagger}\right]\psi. \quad (\text{C1})$$

When the general form for the spin connection in Eq. (2.5) is substituted there are a number of cancellations. To display the result it is convenient to define coefficients which are traceless on their event indices:

$$\bar{t}_{\mu}^{\alpha\beta} = t_{\mu}^{\alpha\beta} - \frac{1}{3}(\delta_{\mu}^{\alpha}t_{\lambda}^{\lambda\beta} - \delta_{\mu}^{\beta}t_{\lambda}^{\lambda\alpha}) \quad (\text{C2a})$$

$$\bar{v}_{\mu}^{\alpha} = v_{\mu}^{\alpha} - \frac{1}{4}\delta_{\mu}^{\alpha}v_{\lambda}^{\lambda} \quad (\text{C2b})$$

$$\bar{a}_{\mu}^{\alpha} = a_{\mu}^{\alpha} - \frac{1}{4}\delta_{\mu}^{\alpha}a_{\lambda}^{\lambda}. \quad (\text{C2c})$$

Then the matrix occurring in the action can be written

$$h\gamma^{\mu}\Gamma_{\mu} - \left(h\gamma^{\mu}\Gamma_{\mu}\right)^{\dagger} = h(\gamma^{\mu}A_{\mu} + B), \quad (\text{C3})$$

where the matrix  $A_{\mu}$  contains the “traceless” part of the coefficients,

$$\begin{aligned} A_{\mu} = & 2(\text{Re}\bar{t}_{\mu}^{\alpha\beta})[\gamma_{\alpha}, \gamma_{\beta}] + 2(\text{Re}\bar{v}_{\mu}^{\alpha})\gamma_{\alpha} \\ & + 2i(\text{Im}\bar{a}_{\mu}^{\alpha})\gamma_{\alpha}\gamma_5 + 2i(\text{Im}p_{\mu})\gamma_5 + 2i(\text{Im}s_{\mu})I, \end{aligned} \quad (\text{C4a})$$

and the matrix  $B$  contains the “traces”:

$$B = 8i(\text{Im}t_{\lambda}^{\lambda\beta})\gamma_{\beta} + 2i(\text{Im}v_{\lambda}^{\lambda})I + 2(\text{Re}a_{\lambda}^{\lambda})\gamma_5. \quad (\text{C4b})$$

The full spin connection in Eq. (2.5) contains 16 complex or 32 real parameters for a fixed value of  $\mu$ . The combination that occurs in the action contains 16 real parameters for a fixed  $\mu$ .

### APPENDIX D: SECOND ORDER DIRAC EQUATION AND THE SPIN CURVATURE

If one iterates the Dirac equation (3.5a) the result is a second-order equation

$$0 = g^{\mu\nu}\hat{\nabla}_{\mu}\hat{\nabla}_{\nu}\psi + m^2\psi + \frac{1}{2}[\gamma^{\mu}, \gamma^{\nu}]\nabla_{\mu}\nabla_{\nu}\psi, \quad (\text{D1})$$

where the second order covariant derivative is

$$\nabla_{\mu}\nabla_{\nu}\psi = \partial_{\mu}(\nabla_{\nu}\psi) - \Gamma_{\mu\nu}^{\lambda}(\nabla_{\lambda}\psi) + \Gamma_{\mu}(\nabla_{\nu}\psi).$$

In the second term of Eq. (D1) the antisymmetric combination of covariant derivatives defines the spin curvature tensor  $\Phi_{\mu\nu}$ :

$$\nabla_{\mu}\nabla_{\nu}\psi - \nabla_{\nu}\nabla_{\mu}\psi = \Phi_{\mu\nu}\psi,$$

where

$$\Phi_{\mu\nu} = \partial_{\mu}\Gamma_{\nu} - \partial_{\nu}\Gamma_{\mu} + \Gamma_{\mu}\Gamma_{\nu} - \Gamma_{\nu}\Gamma_{\mu}. \quad (\text{D2})$$

Since  $\Gamma_{\mu}$  is a linear combination of the Clifford algebra matrices  $[\gamma_{\alpha}, \gamma_{\beta}]$ ,  $\gamma_{\alpha}$ ,  $\gamma_{\alpha}\gamma_5$ ,  $\gamma_5$ , and  $I$  one would expect that  $\Phi_{\mu\nu}$  also contains these matrices. (The part of the spin connection that is proportional to the identity matrix will trivially cancel in  $\Phi_{\mu\nu}$  so that  $\Gamma_{\mu}$  and  $\hat{\Gamma}_{\mu}$  produce the same spin curvature tensor.)

The spin curvature tensor can be related to the Riemann-Christoffel curvature tensor

$$R_{\mu\nu\kappa}^{\cdot\cdot\cdot\lambda} = \partial_\mu \Gamma_{\nu\kappa}^\lambda - \partial_\nu \Gamma_{\mu\kappa}^\lambda + \Gamma_{\mu\alpha}^\lambda \Gamma_{\nu\kappa}^\alpha - \Gamma_{\nu\alpha}^\lambda \Gamma_{\mu\kappa}^\alpha, \quad (D3)$$

in the notation of Schouten [15]. The commutator of two covariant derivatives acting on a vector field is

$$\nabla_\mu \nabla_\nu A^\lambda - \nabla_\nu \nabla_\mu A^\lambda = R_{\mu\nu\kappa}^{\cdot\cdot\cdot\lambda} A^\kappa.$$

By working out  $\nabla_\mu \nabla_\nu \gamma^\lambda - \nabla_\nu \nabla_\mu \gamma^\lambda = 0$  one obtains the relation

$$0 = R_{\mu\nu\kappa}^{\cdot\cdot\cdot\lambda} \gamma^\kappa + [\Phi_{\mu\nu}, \gamma^\lambda]. \quad (D4)$$

Since  $\Phi_{\mu\nu}$  contains no identity component, this equation requires the spin curvature to be entirely in the Lorentz subalgebra:

$$\Phi_{\mu\nu} = -\frac{1}{8} R_{\mu\nu\alpha\beta} [\gamma^\alpha, \gamma^\beta]. \quad (D5)$$

When this is substituted into Eq. (D1) the second order form of the Dirac equation becomes

$$0 = g^{\mu\nu} \widehat{\nabla}_\mu \widehat{\nabla}_\nu \psi + m^2 \psi + (i\gamma_5 \tilde{R} - \frac{1}{4} IR) \psi \quad (D6)$$

where

$$\begin{aligned} \tilde{R} &= \frac{1}{8\sqrt{-g}} \epsilon^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta} \\ R &= g^{\mu\beta} g^{\nu\alpha} R_{\mu\nu\alpha\beta}. \end{aligned}$$

## APPENDIX E: SPIN BASE TRANSFORMATION TO VIERBEIN BASE

According to the theorem proven in Appendix B, any two sets of Dirac matrices that solve the anticommutation relations must be related by a spin-base similarity transformation. Consequently any solution  $\gamma^\mu$  to the anticommutation relations is spin-base equivalent to the vierbein solution:

$$\gamma^\mu = S e_a^\mu \gamma^a S^{-1}. \quad (E1)$$

This Appendix shows how to construct a spin base transformation  $S$  which does this.

1. Starting with any set  $\gamma^\mu$ , compute

$$\gamma_5(x) = -i \frac{\sqrt{-g}}{4!} \epsilon_{\alpha\beta\mu\nu} \gamma^\alpha \gamma^\beta \gamma^\mu \gamma^\nu. \quad (E2)$$

Since  $[\gamma_5(x)]^2 = I$  the eigenvalues of  $\gamma_5(x)$  are  $\pm 1$ . It is elementary to find a matrix  $S_1$  that diagonalizes  $\gamma_5(x)$  and therefore transforms it to a constant matrix. Is important that the transformed matrix be constant but it need not be diagonal for what follows. Thus let

$$\gamma_{(5)} = S_1^{-1} \gamma_5(x) S_1 \quad (E3)$$

where  $\gamma_{(5)}$  is a constant matrix chosen in some convenient form. Associated with this constant matrix are a set of constant Dirac matrices  $\gamma^a$  which satisfy  $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$  in a representation such that  $i\gamma^{(0)}\gamma^{(1)}\gamma^{(2)}\gamma^{(3)} = \gamma_{(5)}$ . From  $S_1$  and the original Dirac matrices construct the set

$$\gamma'^\mu(x) = S_1^{-1} \gamma^\mu(x) S_1. \quad (E4)$$

Each of the new matrices  $\gamma'^\mu(x)$  can be written as a linear combination of the 15 constant matrices  $\gamma^a$ ,  $\gamma^a \gamma_{(5)}$ ,  $\gamma_{(5)}$ , and  $[\gamma^a, \gamma^b]$ . However the vanishing anticommutator

$$\{\gamma_{(5)}, \gamma'^\mu(x)\} = 0$$

means that each  $\gamma'^\mu(x)$  is actually a linear combination only of the 8 constant matrices  $\gamma^a$  and  $\gamma^a \gamma_{(5)}$ . Thus write

$$\gamma'^\mu(x) = V_a^\mu \gamma^a + A_a^\mu \gamma^a \gamma_{(5)}. \quad (E5)$$

The anticommutator of these Dirac matrices is

$$\begin{aligned} \{\gamma'^\mu(x), \gamma'^\nu(x)\} &= 2V_a^\mu V_b^\nu \eta^{ab} - 2A_a^\mu A_b^\nu \eta^{ab} \\ &\quad + (V_a^\mu A_b^\nu + V_a^\nu A_b^\mu) [\gamma^a, \gamma^b] \gamma_{(5)} \end{aligned}$$

Since the anticommutator equals  $2g^{\mu\nu}$ , it follows that

$$\begin{aligned} g^{\mu\nu} &= V_a^\mu V_b^\nu \eta^{ab} - A_a^\mu A_b^\nu \eta^{ab} \\ 0 &= (V_a^\mu A_b^\nu + V_a^\nu A_b^\mu) [\gamma^a, \gamma^b] \gamma_{(5)} \end{aligned} \quad (E6)$$

The second condition can only be satisfied if  $A_a^\mu$  is proportional to  $V_a^\mu$ . Therefore set

$$A_a^\mu = V_a^\mu \tanh \theta$$

where  $\theta$  is some function of space-time. Eq. (E6) becomes

$$g^{\mu\nu} = V_a^\mu V_b^\nu \eta^{ab} (1 - \tanh^2 \theta)$$

Thus define the vierbein

$$e_a^\mu = V_a^\mu / \cosh \theta,$$

It automatically satisfies

$$g^{\mu\nu} = e_a^\mu e_b^\nu \eta^{ab} \quad (E7)$$

The Dirac matrices in Eq. (E5) are now

$$\gamma'^\mu = E_a^\mu \gamma^a (I \cosh \theta + \gamma_{(5)} \sinh \theta)$$

Since the matrices  $\gamma^{(0)} \gamma^a$  and  $\gamma^{(0)} \gamma^a \gamma_{(5)}$  are self-adjoint, the matrices  $\gamma^{(0)} \gamma'^\mu$  will only be self-adjoint if the vierbein  $e_a^\mu$  and the function  $\theta$  are real.

2. To eliminate the  $\theta$ -dependent chiral rotation, define another similarity transformation

$$S_2 = I \cosh(\theta/2) - \gamma_{(5)} \sinh(\theta/2).$$

This matrix has the property

$$\gamma^a(I \cosh \theta + \gamma_{(5)} \sinh \theta) = S_2 \gamma^a S_2^{-1}.$$

Therefore

$$\gamma'^\mu = S_2 E_a^\mu \gamma^a S_2^{-1} \quad (\text{E8})$$

3. Combing Eqs. (E4) and (E8) shows that the original space-time dependent Dirac matrices can be expressed as

$$\gamma^\mu = S_1 S_2 E_a^\mu \gamma^a S_2^{-1} S_1^{-1} \quad (\text{E9})$$

The above procedure gives an explicit method for constructing the required similarity transformations.

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